CS 208: Automata Theory and Logic

Lecture 4: Regular Expressions and Finite Automata

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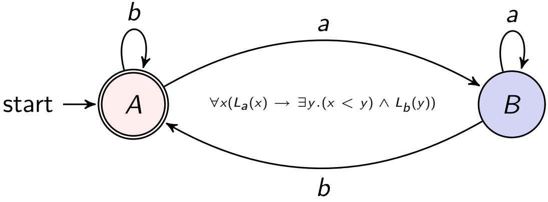

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- Definitions of the factorial function and Fibonacci sequence
- Definition of a singly-linked list in Java:

```
class List <E> {
  E value; \* Base case *\
  List<E>  next; \* Induction *\ }
```

- Definition of a tree

Structural Induction

Definition (Principle of Structural Induction)

Prove an assertion about a set S recursively defined using a set X given in the basis, and a set of rules using $s_1, s_2, \ldots, s_k \in S$ for producing new members of the set.

- (Base case): Prove the assertion for every element in X.
- (Induction): Assume that the assertion holds for arbitrarily chosen $s_1, s_2, \ldots, s_k$ and using this fact prove that all elements of S produced using the recursive definition and $s_1, s_2, \ldots, s_k$ satisfy the assertion.

Examples:

- For all $n \geq 0$ we have that $\sum_{i=0}^{n} i = n(n+1)/2$.
- Every expression defined has an equal number of left and right parenthesis.
- Every tree has one more node than the edges.
- Other examples

What are Regular Languages?

- An alphabet $\Sigma = \{a, b, c\}$ is a finite set of letters,
- The set of all strings (aka, words) Σ^* over an alphabet Σ can be recursively defined as: as :
  - Base case: $\varepsilon \in \Sigma^*$ (empty string),
  - Induction: If $w \in \Sigma^*$ then $wa \in \Sigma^*$ for all $a \in \Sigma$.
- A language L over some alphabet Σ is a set of strings, i.e. $L \subseteq \Sigma^*$.
- Some examples:
  - $L_{\mathrm{even}} = \{w \in \Sigma^* : w \text{ is of even length}\}$
  - $L_{a^*b^*} = \{w \in \Sigma^* : w \text{ is of the form } a^n b^m \text{ for } n, m \geq 0\}$
  - $L_{a^n b^n} = \{w \in \Sigma^* : w \text{ is of the form } a^n b^n \text{ for } n \geq 0\}$
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- Deterministic finite state automata define languages that require finite resources (states) to recognize.

Definition (Regular Languages)

We call a language regular if it can be accepted by a deterministic finite state automaton.

Why they are "Regular"

- A number of widely different and equi-expressive formalisms precisely capture the same class of languages:
  - Deterministic finite state automata
  - Nondeterministic finite state automata (also with ε-transitions)
  - Kleene's regular expressions, also appeared as Type-3 languages in Chomsky's hierarchy [Cho59].
  - Monadic second-order logic definable languages [B6̈0, Elg61, Tra62]
  - Certain Algebraic connection (acceptability via finite semi-group) [RS59]

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We have already seen that:

Theorem (DFA=NFA=ε-NFA)

A language is accepted by a deterministic finite automaton if and only if it is accepted by a non-deterministic finite automaton.

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In this lecture we introduce Regular Expressions, and prove:

Theorem (REGEX=DFA)

A language is accepted by a deterministic finite automaton if and only if it is accepted by a regular expression.

Regular Expressions (RegEx)

- textual (declarative) way to represent regular languages (compare automata)
- Users of UNIX-based systems will already be familiar with these expressions:
  - **ls lecture*.pdf**
  - **rm -rf *.***
  - **grep automat* /usr/share/dict/words**
  - Also used in AWK, expr, Emacs and vi searches,
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- Some useful String-set operations:
  - union $L \cup M \stackrel{\text{def}}{=} \{w : w \in L \text{ or } w \in M\}$
  - concatenation $L.M \stackrel{\text{def}}{=} \{u.v : u \in L \text{ and } v \in M\}$
  - self-concatenation let $L^2 \stackrel{\text{def}}{=} L.L$, similarly $L^3, L^4, \ldots$. Also $L^0 \stackrel{\text{def}}{=} \{\varepsilon\}$.
  - S. C. Kleene cite proposed notation L^* to denote closure of self-concatenation operation, i.e. $L^* \stackrel{\text{def}}{=} \cup_{i \geq 0} L^i$.
  - Examples $L = \{\varepsilon\}$ and $L = \{0, 1\}$

Regular Expressions: Inductive Definition

For a regular expression E we write $L(E)$ for its language. The set of valid regular expressions *RegEx* can be defined recursively as the following:

	Syntax	Semantics
(empty string)	$\varepsilon \in RegEx$	$L(\varepsilon) = \{\varepsilon\}$
(*empty set*)	$\emptyset \in RegEx$	$L(\emptyset) = \emptyset$
(*single letter*)	$\mathbf{a} \in RegEx$	$L(\mathbf{a}) = \{a\}$
(*variable*)	$L \in RegEx$	where L is a language variable.
(*union*)	$E + F \in RegEx$	$L(E + F) = L(E) \cup L(F)$
(*concatenation*)	$E.F \in RegEx$	$L(E.F) = L(E).L(F)$
(*Kleene Closure*)	$E^* \in RegEx$	$L(E^*) = (L(E))^*$
(*Parenthetic Expression*)	$(E) \in RegEx$	$L((E)) = L(E)$.

Precedence Rules: $* > . > +$

Example : $01^* + 1^*0^* \stackrel{\text{def}}{=} (0.(1^*)) + ((1^*).(0^*))$

Regular Expressions: Examples

Find regular expressions for the following languages:

- The set of all strings with an even number of 0's
- The set of all strings of even length (length multiple of k)
- The set of all strings that begin with 110
- The set of all strings containing exactly three 1's
- The set of all strings divisible by 2
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- Practice writing regular expressions for the languages accepted by finite state automata.
- Can we generalize this intuitive construction?
- Can we construct a DFA/NFA for a regular expression?

Finite Automata to Regular Expressions

Theorem

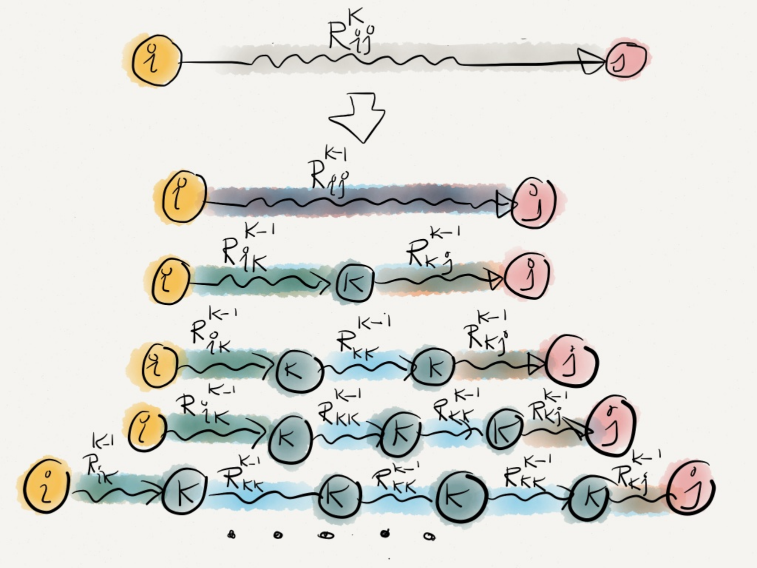
For every deterministic finite automaton A there exists a regular expression E_A such that $L(A) = L(E_A)$.

Proof.

- Let states of automaton A be $\{1, 2, \ldots, n\}$.
- Consider $R_{i,j}^{(k)}$ be the regular expression whose language is the set of labels of path from i to j without visiting any state with label larger than k.
- (Basis): $R_{i,j}^{(0)}$ collects labels of direct paths from i to j,
  - $R_{i,j}^{(0)} = a_1 + a_2 + \cdots + a_n$ if $\delta(i, a_k) = j$ for $1 \leq k \leq n$
  - if $i = j$ then it also includes ε.
- (Induction): Compute $R_{i,j}^{(k)}$ using $R_{i,j}^{(k-1)}$'s.

$\square$

Computing $R_{i,j}^{(k)}$ using $R_{i,j}^{(k-1)}$



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- (Induction): Compute $R_{i,j}^{(k)}$ using $R_{i,j}^{(k-1)}$'s.

$$R_{i,j}^{(k)} = R_{i,j}^{(k-1)} + R_{i,k}^{(k-1)}.(R_{k,k}^{(k-1)})^*.R_{k,j}^{(k-1)}.$$

- E_A is $R_{i_0,f_1}^{(n)} + R_{i_0,f_2}^{(n)} + \cdots + R_{i_0,f_k}^{(n)}$.

Alternative Method–Eliminating States

Shortcomings of previous reduction:

- The previous method works in all the settings, but is expensive (up to n^3 expressions, with a factor of 4 blowup in each step).
- For each i, j, i', j', both $R_{i,j}^{(k+1)}$ and $R_{i',j'}^{(k+1)}$ store expression $(R_{k,k}^{(k)})^*$. This duplication can be avoided.

Alternative (more intuitive) method:

- A "beast" in the middle: Finite automata with regular expressions
- Remove all states except final and initial states in an "intuitive" way.
- Trivial to write regular expressions for DFA with only two states: an initial and a final one.
- The regular expression is union of this construction for every final state.
- Example

figure2

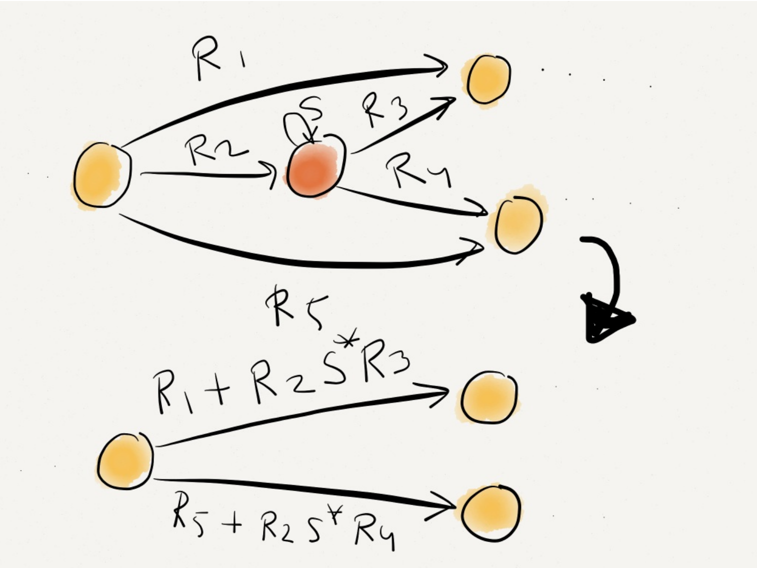
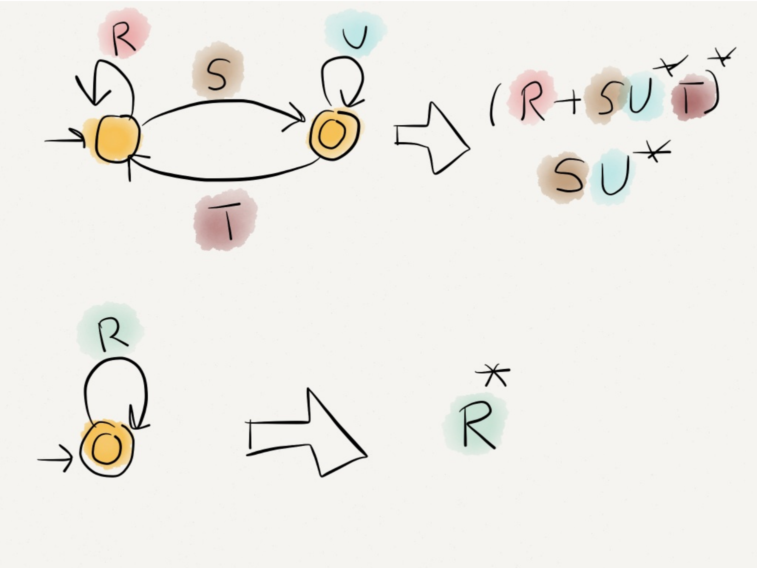


figure3



Regular Expressions to Finite Automata

Theorem

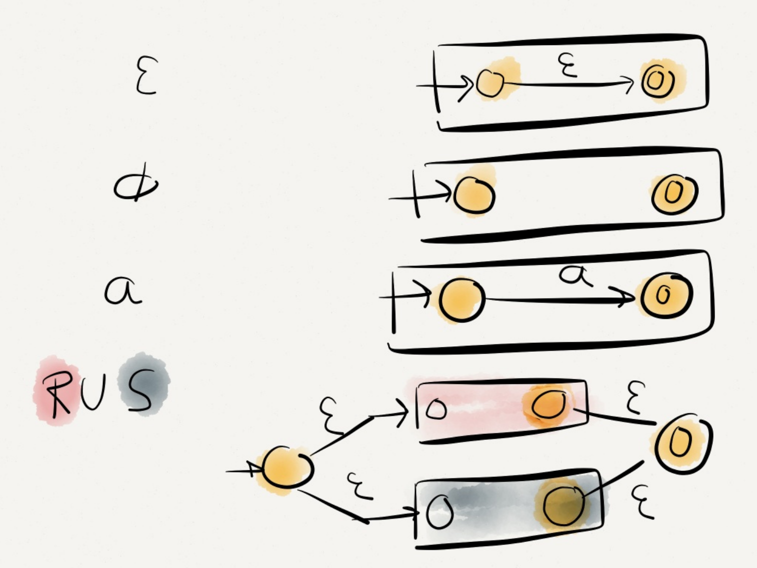
For every regular expression E there exists a deterministic finite automaton A_E such that $L(E) = L(A_E)$.

Proof.

- Via induction on the structure of the regular expressions we show a reduction to nondeterministic finite automata with ε-transitions.
- Result follows form the equivalence of such automata with DFA.

□

Regular Expressions to Finite Automata



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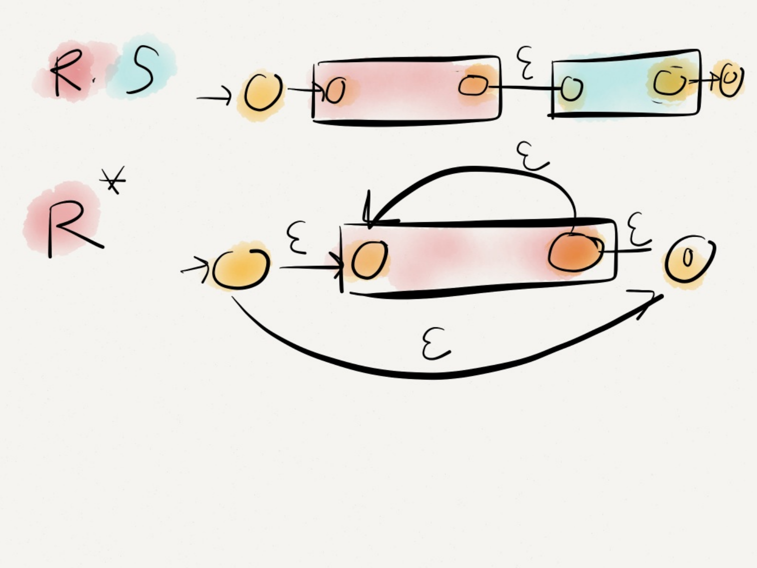


Syntactic Sugar for Regular Expressions in Unix

$$\begin{array}{rcl}
[a_1 a_2 a_3 \ldots a_k] & \text{for} & a_1 + a_2 + \cdots + a_k \\
. & \text{for} & a + b + \cdots + z + A + \ldots \\
| & \text{for} & + \\
R\{5\} & \text{for} & RRRRR \\
R+ & \text{for} & \cup_{i \geq 1} R\{i\} \\
R? & \text{for} & \varepsilon + R
\end{array}$$

Also [A-za-z0-9] ,[:digits:], etc.

Applications:
Check the man page of "grep" (regular expression based search tool) and "lex" (A tool to generate regular expressions based pattern matching tool) to learn more about regular expressions on UNIX based systems.

Algebraic Laws for Regular Expressions

Associativity:

- $L + (M + N) = (L + M) + N$ and $L.(M.N) = (L.M).N$.

Commutativity:

- $L + M = M + L$. However, $L.M \neq M.L$ in general.

Identity:

- $\emptyset + L = L + \emptyset = L$ and $\varepsilon.L = L.\varepsilon = L$

Annihilator:

- $\emptyset.L = L.\emptyset = \emptyset$

Distributivity:

- left distributivity $L.(M + N) = L.M + L.N$.
- right distributivity $(M + N).L = M.L + N.L$.

Idempotent $L + L = L$.

Closure Laws:

- $(L^*)^* = L^*$, $\emptyset^* = \varepsilon$, $\varepsilon^* = \varepsilon$, $L^+ = LL^* = L^*L$, and $L^* = L^+ + \varepsilon$.

DeMorgan Type Law: $(L + M)^* = (L^*M^*)^*$

Verifying laws for regular expressions

Theorem

- *Let E is some regular expressions with variables $L_1 L_2, \ldots, L_m$.*
- *Let C be a regular expression where each L_i is concretized to some letters $a_1 a_2, \ldots a_m$.*
- *Then every string w in $L(E)$ can be written as $w_1 w_2 \ldots w_k$ where w_i is in some language L_{j_i} and $a_{j_1} a_{j_2} \ldots a_{j_k}$ is in $L(C)$.*
- *In other words , the set $L(E)$ can be constructed by taking strings $a_{j_1} a_{j_2} \ldots a_{j_k}$ from $L(C)$ and replacing a_{j_i} with L_{j_i}.*

Proof.

A simple induction over the structure of regular expression E. □

Example

Theorem (Application)

Proof of a concretized law carries over to abstract law.

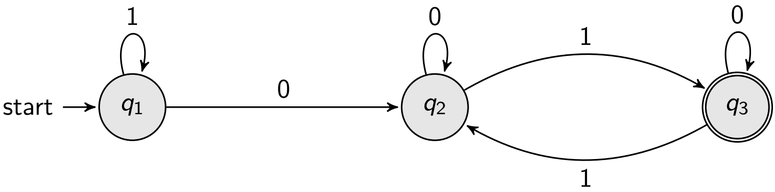
Example

Prove that $(\varepsilon + L)^* = L^*$.
We can concretize the rule as $(\varepsilon + a)^* = a^*$. Let's prove the concretized law, and we know that the result will carry over to the abstract law.

$$\begin{aligned}(\varepsilon + a)^* &= (\varepsilon^*.a^*)^* \\ &= (\varepsilon.a^*)^* \\ &= (a^*)^* \\ &= a^*.\end{aligned}$$

First equality holds since $(L + M)^* = (L^*.M^*)^*$. The second equality holds since $\varepsilon^* = \varepsilon$. The third equality holds as ε is identity for concatenation, while the last equality follows from $(L^*)^* = L^*$.

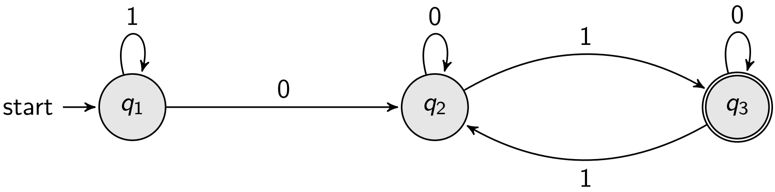
Example



	$R_{1,1}$	$R_{1,2}$	$R_{1,3}$	$R_{2,1}$	$R_{2,2}$	$R_{2,3}$	$R_{3,1}$	$R_{3,2}$	$R_{3,3}$
(0)	$1+\varepsilon$	0	$\emptyset$	$\emptyset$	$0+\varepsilon$	1	$\emptyset$	1	$0+\varepsilon$
(1)	1^*	1^*0	$\emptyset$	$\emptyset$	$0+\varepsilon$	1	$\emptyset$	1	$0+\varepsilon$
(2)	1^*	1^*00^*	1^*00^*1	$\emptyset$	0^*	0^*1	$\emptyset$	10^*	$(0+\varepsilon)+10^*1$

$$
\begin{aligned}
R_{1,1}^{(1)} &= R_{1,1}^{(0)} + R_{1,1}^{(0)}(R_{1,1}^{(0)})^* R_{1,1}^{(0)} \\
&= (1+\varepsilon) + (1+\varepsilon)(1+\varepsilon)^*(1+\varepsilon) \\
&= (1+\varepsilon)\varepsilon + (1+\varepsilon)(1+\varepsilon)^*(1+\varepsilon) \\
&= (1+\varepsilon)\varepsilon + (1+\varepsilon)1^*(1+\varepsilon) \\
&= (1+\varepsilon)(\varepsilon + 1^*(1+\varepsilon)) = (1+\varepsilon)(\varepsilon + 1^*1 + 1^*\varepsilon) \\
&= (1+\varepsilon)(\varepsilon + 1^+ + 1^*) = (1+\varepsilon)(1^* + 1^*) = (1+\varepsilon)1^* \\
&= 11^* + 1^* = 1^+ + 1^* = 1^+ + 1^+ + \varepsilon = 1^+ + \varepsilon = 1^*
\end{aligned}
$$

Example



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(0)	$1+\varepsilon$	0	$\emptyset$	$\emptyset$	$0+\varepsilon$	1	$\emptyset$	1	$0+\varepsilon$
(1)	1^*	1^*0	$\emptyset$	$\emptyset$	$0+\varepsilon$	1	$\emptyset$	1	$0+\varepsilon$
(2)	1^*	1^*00^*	1^*00^*1	$\emptyset$	0^*	0^*1	$\emptyset$	10^*	$(0+\varepsilon)+10^*1$

$$
\begin{aligned}
R_{1,3}^{(3)} &= R_{1,3}^{(2)} + R_{1,3}^{(2)}(R_{3,3}^{(2)})^* R_{3,3}^{(2)} \\
&= 1^*00^*1 + 1^*00^*1(0+\varepsilon+10^*1)^*(0+\varepsilon+10^*1) \\
&= 1^*00^*1\varepsilon + 1^*00^*1(0+\varepsilon+10^*1)^*(0+\varepsilon+10^*1) \\
&= 1^*00^*1(\varepsilon + (0+\varepsilon+10^*1)^*(0+\varepsilon+10^*1)) \\
&= 1^*00^*1(\varepsilon + (0+10^*1)^*(0+\varepsilon+10^*1)) \\
&= 1^*00^*1(\varepsilon + (0+10^*1)^+ + (0+10^*1)^*) \\
&= 1^*00^*1((0+10^*1)^* + (0+10^*1)^*) = 1^*00^*1(0+10^*1)^*
\end{aligned}
$$

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